Math 1510 Week 3

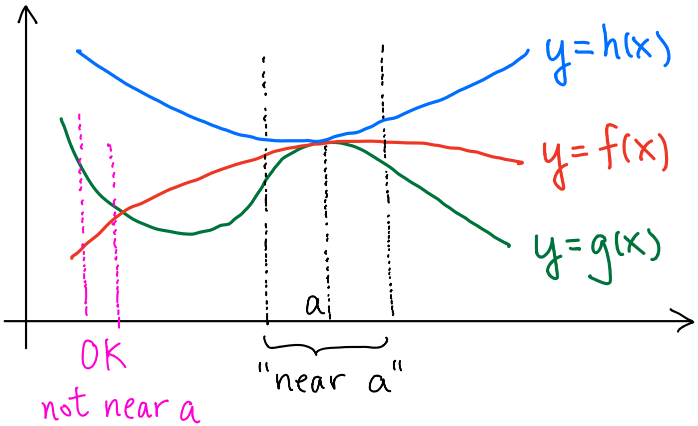
Sandwich Theorem (Squeeze Theorem)

Let $a \in \mathbb{R}$ or $\pm\infty$. Suppose

① $g(x) \le f(x) \le h(x)$ near a

② $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L \in \mathbb{R}$

Then $\lim_{x\to a} f(x) = L$



Rmk Similar result for one-sided limit, and limit of sequence.

eg1 Let $a_n = \dfrac{n+(-1)^n}{2n}$. $\lim_{n\to\infty} a_n = ?$

Sol $\dfrac{n-1}{2n} \le a_n \le \dfrac{n+1}{2n}$

$$\lim_{n\to\infty} \frac{n-1}{2n} = \lim_{n\to\infty} \frac{1-\frac{1}{n}}{2} = \frac{1-0}{2} = \frac{1}{2}$$

$$\lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2} = \frac{1+0}{2} = \frac{1}{2}$$

$$\therefore \lim_{n\to\infty} \frac{n-1}{2n} = \lim_{n\to\infty} \frac{n+1}{2n}$$

By Sandwich Theorem, $\lim_{n\to\infty} a_n = \frac{1}{2}$

eg2

$$\lim_{x\to 0} x\cos\frac{1}{x} = ?$$

Idea: $\lim_{x\to 0} x = 0$, $\cos\frac{1}{x}$ is bounded

Sol For $x \neq 0$,

$$\left|\cos\frac{1}{x}\right| \leq 1$$

$$\left|x\cos\frac{1}{x}\right| \leq |x|$$

$$\Rightarrow \quad -|x| \leq x\cos\frac{1}{x} \leq |x|$$

Note $\lim_{x\to 0} -|x| = 0 = \lim_{x\to 0} |x|$

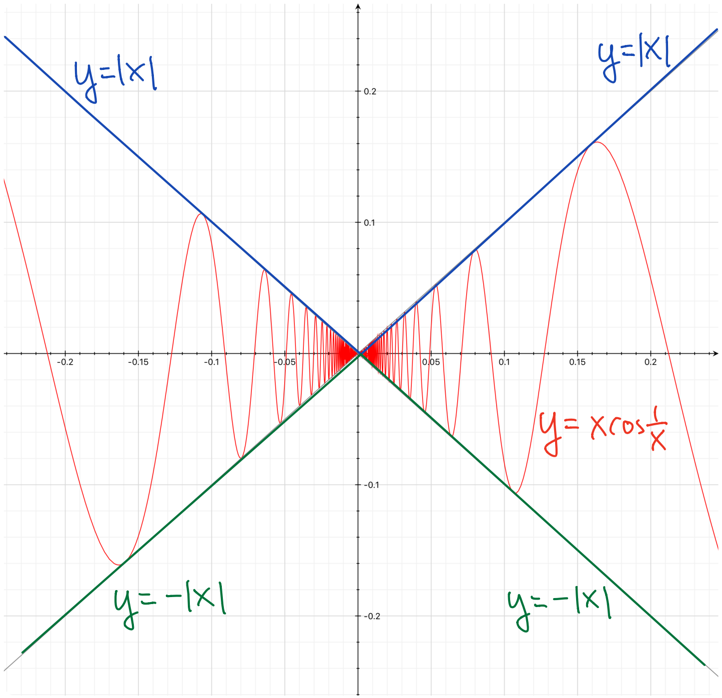
By Sandwich Theorem,

$$\lim_{x\to 0} x\cos\frac{1}{x} = 0$$

Rmk Wrong $\because \lim_{x\to 0}\cos\frac{1}{x}$ DNE

$$\lim_{x\to 0} x\cos\frac{1}{x} = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0}\cos\frac{1}{x}\right)$$

$$= (0)\left(\lim_{x\to 0}\cos\frac{1}{x}\right) = 0$$



Limit involving e

Defn/Thm

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \quad \text{(Limit of sequence)}$$

$$= \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \quad \text{(Limit of function)}$$

$$\approx 2.718281$$

Q Why should $\left(1+\frac{1}{n}\right)^n$ converge?

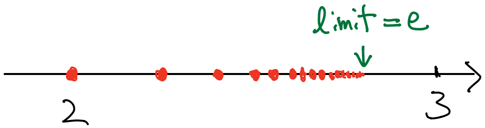
A Monotone Convergence Theorem! (Not for Exam)

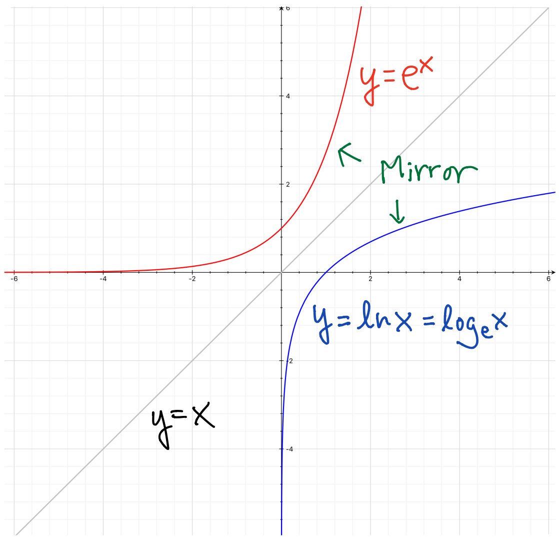
- It is increasing:

  $2,\ 2.25,\ 2.37,\ 2.44,\ 2.49$

- It is bounded above:

  $\left(1+\frac{1}{n}\right)^n < 3$ for $n \in \mathbb{N}$





$$\lim_{x\to\infty} e^x = \infty \quad \longleftrightarrow \quad \lim_{x\to\infty} \ln x = \infty$$

$$\lim_{x\to-\infty} e^x = 0 \quad \longleftrightarrow \quad \lim_{x\to 0^+} \ln x = -\infty$$

eg $\lim_{x\to\infty} e^{2x} - 100e^x \quad (\infty - \infty)$

$$= \lim_{x\to\infty} e^x(e^x - 100)$$

Both factors $\to \infty$

$$= \infty$$

eg

① $\lim_{n\to\infty}\left(1+\frac{1}{5n}\right)^{5n}$

Let $y=5n$.

As $n\to\infty$, $y\to\infty$

$\therefore \lim_{n\to\infty}\left(1+\frac{1}{5n}\right)^{5n}$

$=\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}$

$=e$

Key point

$\lim_{n\to\infty}\left(1+\frac{1}{5n}\right)^{5n}$

same, $\to\infty$

$\Rightarrow$ limit $=e$

② $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{3n}$

$=\lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{2}}\right)^{\frac{n}{2}\cdot 6}$

$=\left[\lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{2}}\right)^{\frac{n}{2}}\right]^{6}$

same, $\frac{n}{2}\to\infty$

$=e^{6}$

③ $\lim_{x\to0^+}(1+x)^{\frac{1}{x}}$

Let $y=\frac{1}{x}$

As $x\to0^+$, $y\to\infty$

$\therefore \lim_{x\to0^+}(1+x)^{\frac{1}{x}}$

$=\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}$

$=e$

④ $\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^{x}$

$=\lim_{x\to\infty}\left(1+\frac{2}{x-1}\right)^{x}$

$=\lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2}\cdot2+1}$

$=\lim_{x\to\infty}\left[\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2}}\right]^{2}\left(1+\frac{1}{\frac{x-1}{2}}\right)$

$=e^{2}(1+0)=e^{2}$

One can show that

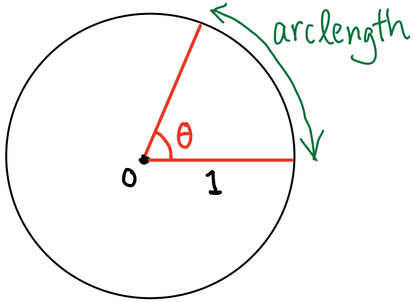
$$\boxed{\begin{gathered}\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x}=e\\ \lim_{x\to0}(1+x)^{\frac{1}{x}}=e\end{gathered}}$$

Rmk $\lim_{x\to\infty}(1+x)^{\frac{1}{x}}=1$

Limits involving Trigonometric functions

Radian (a unit for angles)

Consider a unit circle (radius = 1)



> Define
>
> $\theta = x$ rad
>
> if arclength $= x$

360° = full circle = 2π rad

180° = half circle = π rad

90° = right angle = $\frac{\pi}{2}$ rad

$$\left(\frac{180 x}{\pi}\right)^\circ = x \text{ rad}$$

$$y^\circ = \frac{\pi y}{180} \text{ rad}$$

> We use Radian, NOT degree, in calculus

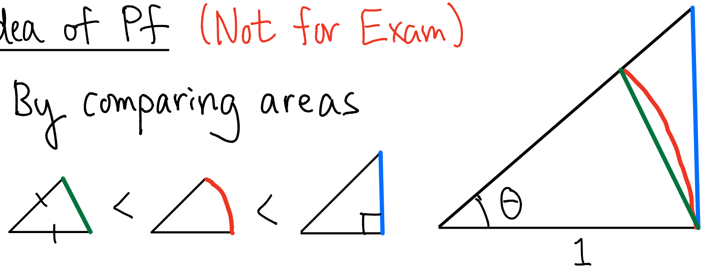
> **Thm** $\displaystyle \lim_{x\to 0} \frac{\sin x}{x} = 1$

Rmk x in $\sin x$ is in radian

Rmk $\displaystyle \lim_{x\to 0} \frac{x}{\sin x} = \frac{1}{\displaystyle \lim_{x\to 0} \frac{\sin x}{x}} = 1$

Idea of Pf (Not for Exam)

- By comparing areas



show $|\sin x| \le |x| \le |\tan x|$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

- Apply Sandwich theorem

eg $\displaystyle \lim_{x\to 0} \frac{\sin 3x}{\sin 4x} = \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot \frac{4x}{\sin 4x}$ same

$= \left(\frac{3}{4}\right)(1)(1)$ As $x \to 0$, $4x \to 0$

$= \frac{3}{4}$

$$\boxed{\underline{\text{Thm}} \quad \lim_{x \to 0} \frac{1-\cos x}{x} = 0}$$

<u>Pf</u>

$$\lim_{x \to 0} \frac{1-\cos x}{x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x}$$

$$= \lim_{x \to 0} \frac{1-\cos^2 x}{x(1+\cos x)}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}$$

$$= (1) \cdot \frac{0}{1+1}$$

$$= 0$$

<u>Ex</u> Prove it by formula $\cos x = 1 - 2\sin^2 \frac{x}{2}$

<u>Ex</u> Find $\lim_{x \to 0} \frac{1-\cos x}{x^2}$

Recall:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

<u>eg</u>

$$\lim_{x \to 0} \frac{\tan x}{x \sec x} = \lim_{x \to 0} \frac{\frac{\sin x}{\cos x}}{x \cdot \frac{1}{\cos x}}$$

$$= \lim_{x \to 0} \frac{\sin x}{x}$$

$$= 1$$

<u>eg</u>

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\pi - 2x} = \lim_{x \to \frac{\pi}{2}} \frac{1}{2} \frac{\sin(\frac{\pi}{2} - x)}{\frac{\pi}{2} - x}$$

$$= \frac{1}{2}(1) \qquad \left(\begin{array}{l}\text{As } x \to \frac{\pi}{2}, \\ \frac{\pi}{2} - x \to 0\end{array}\right)$$

$$= \frac{1}{2}$$

Continuity

Defn $f(x)$ is said to be continuous at a

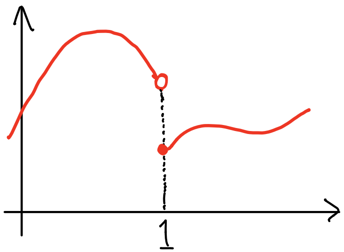
if

$$\underbrace{\lim_{x\to a} f(x)}_{\text{limit exists}} \underset{\text{equal}}{=} \underbrace{f(a)}_{a\in D_f}$$

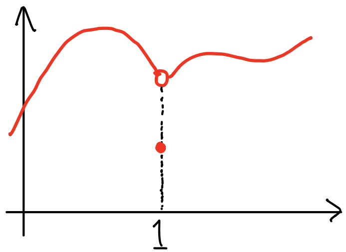
$f(x)$ is said to be a continuous function

if it is continuous at every $a \in D_f$.

eg Discontinuous at 1

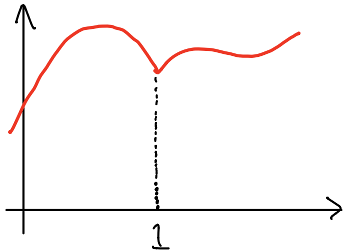


$\lim_{x\to 1} f(x)$ DNE



$\lim_{x\to 1} f(x)$ exists, $\neq f(1)$

eg Continuous at 1



$\lim_{x\to 1} f(x) = f(1)$

Rmk We secretly used continuity when we found limit by substitution.

eg

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} x+1 = 1+1 = 2$$

$\because$ $x+1$ is continuous

eg

Let $f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1 = f(0)$$

$\therefore$ f is continuous at 0.

Some Fact about Continuous functions

① If f, g are continuous at a,

then $f \pm g$, fg, $\frac{f}{g}$ (if $g(a) \neq 0$)

are continuous at a

② Suppose f is continuous at a.

- If $\{a_n\}$ is a sequence with $\lim_{n\to\infty} a_n = a$,

  then $\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(a)$

- If g is a function with $\lim_{x\to b} g(x) = a$,

  then $\lim_{x\to b} f(g(x)) = f\left(\lim_{x\to b} g(x)\right) = f(a)$

If g is continuous at b f is continuous at $g(b)$ then $f \circ g$ is continuous at b.

Examples of continuous functions

- x^a, $|x|$, a^x, $\log_a x$
- $\sin x$, $\cos x$, $\tan x$,

  $\csc x$, $\sec x$, $\cot x$
- Inverse Trigonometric functions (eg. $\arcsin x$)
- Polynomial, Rational functions
- Sum, Difference, Product, Quotient,

  Composition of the functions above.

eg

$$\frac{x^2 + 7x \ln(x^2+1)}{|\cos(e^{\sin x} + \tan x)|}$$

is continuous (on its domain)

eg Find c such that f(x) is continuous

ⓐ $f(x) = \begin{cases} \sqrt{x+1} & \text{if } x \geq 3; \\ x^2 + c & \text{if } x < 3. \end{cases}$

ⓑ $f(x) = \begin{cases} \dfrac{x}{|x|} & \text{if } x \neq 0; \\ c & \text{if } x = 0. \end{cases}$

Sol

ⓐ Clearly, f(x) is continuous for $x \neq 3$

How about at 3 ?

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x^2 + c = 9 + c$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \sqrt{x+1} = \sqrt{3+1} = 2$$

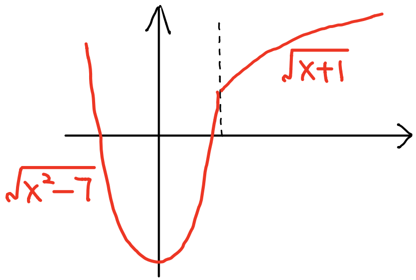
For f(x) to be continuous at 3,

$$\lim_{x \to 3} f(x) = f(3)$$

$$\Rightarrow \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3) = 2$$

$$\Rightarrow 9 + c = 2 = 2$$

$$\Rightarrow c = -7$$



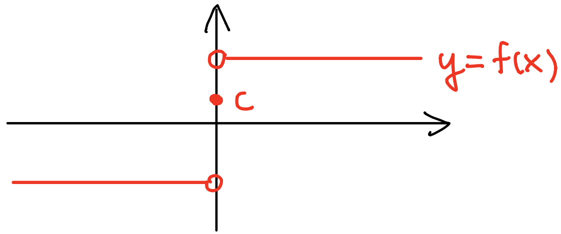
ⓑ If $x < 0$, $f(x) = \dfrac{x}{|x|} = \dfrac{x}{-x} = -1$

If $x > 0$, $f(x) = \dfrac{x}{|x|} = \dfrac{x}{x} = 1$

$$\therefore \lim_{x \to 0^-} f(x) = -1, \quad \lim_{x \to 0^+} f(x) = 1 \neq \lim_{x \to 0^-} f(x)$$

$\Rightarrow \lim_{x \to 0} f(x)$ DNE

$\Rightarrow$ f(x) is not continuous at 0 for any c



eg $\lim_{x\to\infty} \cos\left[\left(1+\frac{1}{2x}\right)^x\right]$

Sol Since cosine is continuous

$$\lim_{x\to\infty} \cos\left[\left(1+\frac{1}{2x}\right)^x\right]$$

$$= \cos\left[\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^x\right]$$

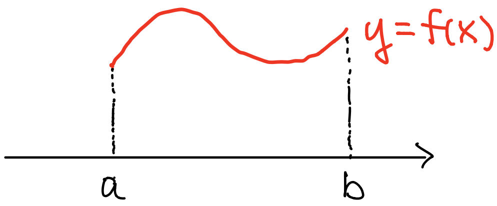
$$= \cos\left[\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^{2x}\right]^{\frac{1}{2}}$$

$$= \cos\sqrt{e}$$

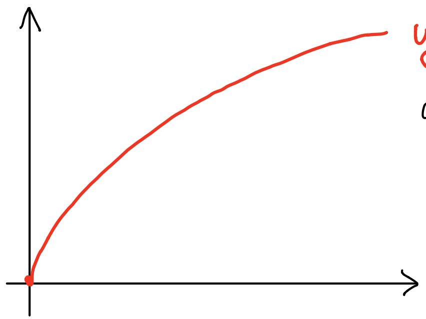
Continuity at endpoints

Let $f:[a,b]\longrightarrow\mathbb{R}$. Then f is said to be

$$\begin{cases} \text{continuous at } a \text{ if } \lim_{x\to a^+} f(x) = f(a) \\ \text{continuous at } b \text{ if } \lim_{x\to b^-} f(x) = f(b) \end{cases}$$



eg



$y=\sqrt{x}$ is continuous on $[0,\infty)$.

Properties of Continuous Functions

Maximum / Minimum

Defn Let $f: A \to \mathbb{R}$, $a \in A$

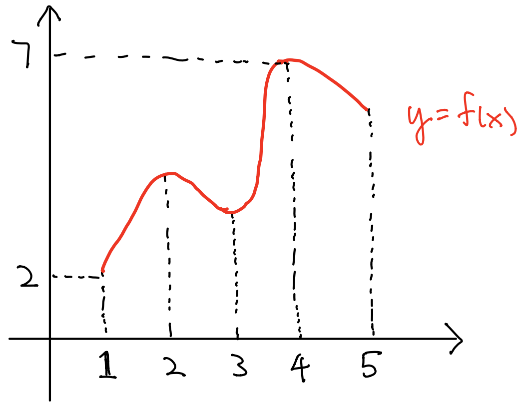
① f is said to have <u>absolute (global) maximum</u> at a if $f(x) \le f(a)$ for all $x \in A$

② f is said to have <u>relative (local) maximum</u> at a if $f(x) \le f(a)$ for all $x \in A$ near a,

③ Similar definitions for <u>absolute (global) minimum</u> and <u>relative (local) minimum</u>

Rmk Absolute max/min is also a relative max/min.

eg Let $f: [1,5] \to \mathbb{R}$



Absolute max at 4

Absolute min at 1

Relative max. at 2, 4

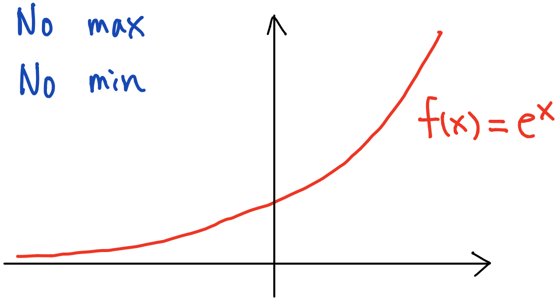
Relative min. at 1, 3, 5

Max Value : 7

Min Value : 2

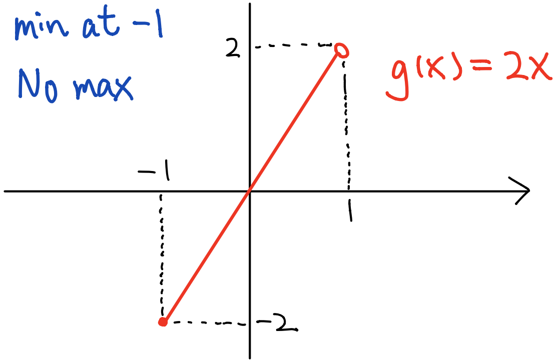
Rmk Not EVERY function has absolute maximum and minimum!

① No max
No min



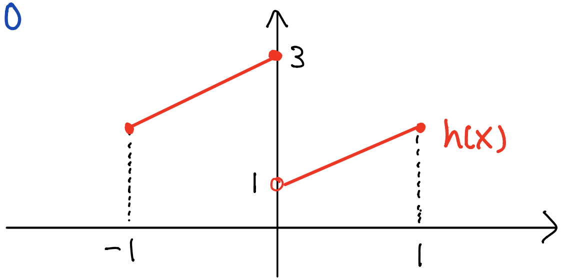
$D_f=(-\infty,\infty)$ is infinitely long

② min at -1
No max



$D_g=[-1,1)$ does not contain all endpoints

③ max at 0
No min.



h is not continuous

Q What functions must have max and min?

> **Extreme Value Theorem (EVT)**
>
> Let $a,b\in\mathbb{R}$, $f:[a,b]\to\mathbb{R}$ be continuous
>
> Then f has both absolute maximum and minimum.

Key points: Domain is closed (contain endpoints) and bounded (finite length), the function is continuous.

Intermediate Value Theorem (IVT)

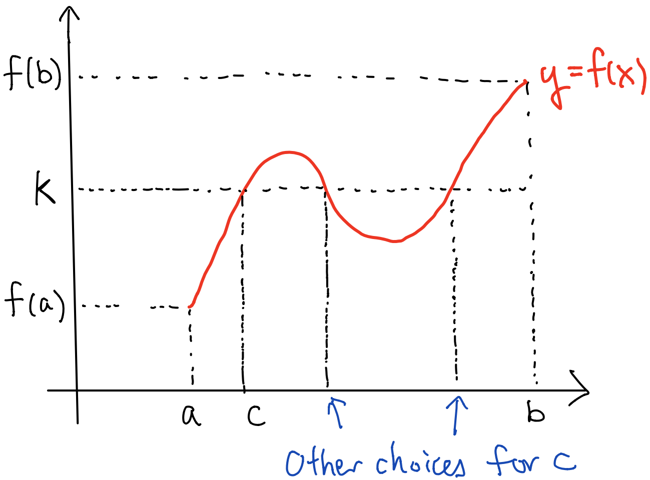
Let $f(x)$ be continuous on $[a,b]$

Suppose $f(a) < k < f(b)$

or $f(a) > k > f(b)$

Then there exists $c \in (a,b)$

such that $f(c) = k$



eg Show that $f(x) = x^6 + 2x^3 - 5$ has at least two real roots

Sol Since $f(x)$ is a polynomial, it is continuous

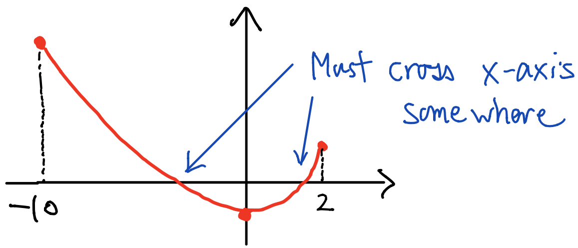
Also, $f(0) = -5 < 0 < f(2) = 75$

By IVT, there exists $c \in (0,2)$ s.t. $f(c) = 0$

Similarly $f(-10) > 0 > f(0)$

By IVT, there exists $d \in (-10, 0)$ s.t. $f(d) = 0$

$\therefore$ f has at least two real roots c, d



Ex Show that

i. any real polynomial of odd degree has a real root.

ii. $x = \cos x$ has a solution. (Hint: Consider $f(x) = x - \cos x$.)